Risk Allocation for Design Optimization with Unidentified Statistical Distributions*

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Reliability based design optimization (RBDO) is often used with specified maximum failure probability for each component, but it can also be used for risk allocation between components. This is done, however under the assumption that the distribution of random variables is known. Design based on regulation (e.g. FAA) allows for the case when distributions are unknown, but it treats components with the same safety margins, not allowing for risk allocation between components. In particular, the regulatory treatment of unknown distribution use non-parametric tolerance intervals to conservatively represent the uncertainty from limited data. One non-parametric tolerance interval methods assumes the true distribution is continuous, while the more restricted Hanson-Koopmans method applies to a class of continuous distributions. The allowable failure strength of a composite material was compared using these non-parametric methods to tolerance intervals assuming a Normal distribution. A simple RBDO problem using the Hanson-Koopmans method was presented to design a UAV wing and horizontal tail to minimize weight, with risk allocation between the two components. The RBDO optimum used a higher effective safety factor on the lighter horizontal tail, which was compared to a design that used the same safety factor on both the tail and wing.

I. Nomenclature

γ	=	Confidence level
σ_a	=	Allowable failure stress
σ_n	=	New failure stress
e	=	Cost function
f	=	Joint probability density function
F_X	=	Cumulative distribution function for random variable X
P	=	Percentile / probability
P_t	=	Tail allowable percentile
P_w	=	Wing allowable percentile
P_f	=	Probability of failure
$\mathbb{P}$	=	Probability set
w_i	=	Initial weight
w_n	=	New weight
w_t	=	Horizontal tail weight
w_w	=	Wing weight
X	=	Random variable
$\boldsymbol{x}$	=	Design variables

II. Introduction

Reliability based design optimization (RBDO) are often used in the literature with specified probability of failure for each component. While this does not perform risk allocation between components, a change in formulation does allow risk allocation, as demonstrated by Acar and Haftka [1]. Regulation-based design, which specifies constant safety

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margins for components and thus does not promote risk allocation between them. This regulation based design is usually called deterministic design because the optimization is performed deterministically, even though the material allowables are determined probabistically.

Acar and Haftka [1] illustrated a safer aircraft design for a configuration made of two panels of substantially unequal weight. The design where risk was allocated between the two panels was safer than a design using equal safety factors for both panels. Both designs used the same total weight. The previous study assumed that the failure stress follows a Normal distribution. Many other RBDO problems presented in literature assume either that the true distribution is known or can be identified [2–5]. Unfortunately, it is difficult to accurately identify the source distribution with limited prior knowledge and sparse data [6, 7].

There have been recent interest in developing distribution free RBDOs to overcome the difficulty associated with identifying statistical distributions. Kanno [8] presented a non-parametric RBDO problem to design a truss with varying loads from an unknown distribution. Ohsaki et al. [9] presented a multi-objective non-parametric RBDO to design a shear frame for all possible robustness levels. The robustness levels can be thought of as non-parametric tolerance intervals (TI) for various percentiles and confidence levels. These non-parametric tolerance intervals are one-order statistics, meaning the TI is determined from one sample within a set of samples. These recent works both utilize non-parametric order statistics to formulate the RBDO problem. The main problems with this method is that available percentile/confidence level combinations depend upon the number of samples, and extrapolation beyond the given worst sample is not possible with this non-parametric method.

This paper will apply a variation of non-parametric order statistics to an RBDO problem similar to [1], where aircraft panels are designed by varying the allowable safety margin for a fixed confidence level. This formulation can be described as using TIs within an RBDO. Construction of TIs is discussed for both known and unknown distributions. The paper goes on to present two different non-parametric methods using order statistics. The first method assumes that the distribution of the true function is continuous (one-order statistic). The other non-parametric method is a two-order statistic, which uses two samples from the sample set to determine a TI. The assumption of the two-order statistic is that the true distribution belongs to the log-concave CDF class of statistical distributions[10–12]. These TIs are then used to determine the allowable failure strength of a composite material.

The paper goes on to present a simple RBDO problem, where the primary uncertainty is due to the allowable failure strength in a composite material. The RBDO problem is applied to the design of an unmanned aerial vehicle (UAV), where the objective is to minimize weight by allocating risk between the wing and horizontal tail. It's required that the probability of failure of the RBDO design must be no worse than a design where the components used the same allowable failure stress. The allowable stresses for the RBDO problem were determined using the Hanson-Koopmans non-parametric method [11]. This method does not assume the failure stress to belong to a single particular distribution, but rather to a class of statistical distributions.

III. RBDO Based on Non-parametric Methods

A typical RBDO can be presented as

$$\text{Minimize: } e(\boldsymbol{x}) \tag{1}$$

$$\text{Subject to: } P_{\text{failure}} \leq P_{\text{required}} \tag{2}$$

where P_{failure} is calculated for known statistical distributions by integrating the joint probability density over the domain

$$P_{\text{failure}} = \int \cdots \int_D f_X(\boldsymbol{x}) d\boldsymbol{x}. \tag{3}$$

There can often be epistemic uncertainty associated with the random quantity X due to the lack of data, and in such cases the RBDO is expressed as

$$\text{Minimize: } e(\boldsymbol{x}) \tag{4}$$

$$\text{Subject to: } \mathbb{P}[P_{\text{failure}} \leq P_{\text{required}}] \geq \gamma \tag{5}$$

where the designer will need to specify the confidence level γ in order to satisfy the constraint.

When there is limited data and the distribution that describes the random variable X is known, then it is possible to numerically calculate whether P_{required} is satisfied to a desired confidence level. In cases when the distribution of a random variable X is unknown, then non-parametric statistics will allow for a provably conservative estimation of whether P_{required} is satisfied. The statistical requirements for the non-parametric methods to be provably conservative are discussed in the next section.

IV. One-Sided Tolerance Intervals for Allowables

A one-sided tolerance interval (TI) can be constructed to represent the allowable failure strength S_a of a material. The TI is expressed as

$$\mathbb{P}[F_X(S_a) \leq P] \geq \gamma \tag{6}$$

for a given percentile P and confidence level γ. The Federal Aviation Administration (FAA) recommends (14 CFR 25.613) different allowable failure strengths deepening upon the redundancy of the design [13]. The 1$^{\text{st}}$ percentile with 95% confidence (A-basis) is used in designs where failure of the single member would result in the loss of structural integrity. The 10$^{\text{th}}$ with 95% confidence (B-Basis) is used in designs of redundant structures, in which failure of a single member would result in the load being safely distributed on the remaining members.

An illustration depicting the process of finding the allowable stress of a material from a sample of tensile tests is shown in Fig. 1. Finding the 10$^{\text{th}}$ to 95% confidence can be thought of as first constructing a distribution of possible 10$^{\text{th}}$ percentiles. The 5$^{\text{th}}$ percentile of this distribution represents the allowable failure stress to 95% confidence. When the true distribution is known, then the TI can be numerically calculated to a desired precision.

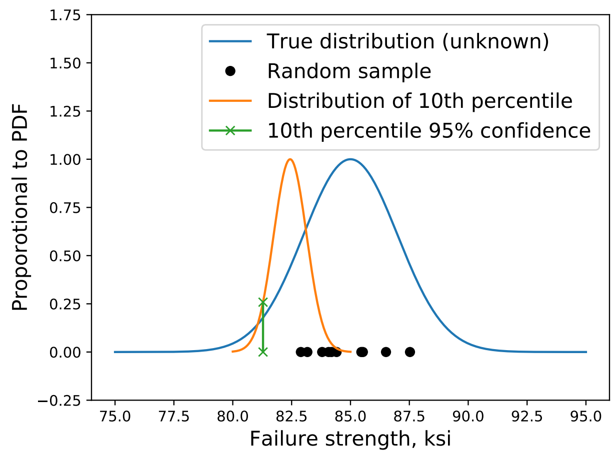


Fig. 1 Illustration of approximating the 10$^{\text{th}}$ percentile with 95% confidence from a random sample which comes from an unknown distribution.

The one-order statistic non-parametric method was used in the RBDO problems by Kanno [8] and [9]. The underlying assumption required by this one-order statistic method is that the true distribution is continuous [14, 15]. A binomial distribution is used in order to conservatively bound the TI for any continues distribution. Unfortunately the one-order statistic is incapable of extrapolating outside of a discrete sample set, which means that calculation of an A-basis or B-basis may require a large number of expensive samples. The smallest percentile (for a lower tail probability) depends on the total number of samples. For example the minimum number of samples required to estimate the A-basis is 299, in which case the allowable is the smallest of the 299 samples [10, 16]. The single sample which represents the TI is the reason the method is called a one-order statistic. Another useful relationship to remember is that estimating the B-basis requires at least 29 samples, where the B-basis occurs from the smallest of the 29 samples.

Hanson and Koopmans [11] derived a proof for non-parametric TIs in order to remedy the shortcomings of the one-order statistic method. The Hanson-Koopmans non-parametric method can determine a TI for arbitrary percentile and confidence level using two samples from the set (e.g. a two-order statistic). The Hanson-Koopmans two-order statistic is capable of extrapolating beyond the smallest sample, while extrapolation is not possible with one-order statistics. Two samples are the minimum required for the Hanson-Koopmans method to estimate any percentile/confidence TI. The underlying assumption is that the true distribution belongs to the class of distributions which have a logarithmically concave (log-concave) cumulative distribution function (CDF). The log-concave CDF class of distributions includes all Normal, Exponential, Gumbel, Laplace, Logistic, Rayleigh, Maxwell, Uniform, Lognormal, and Pareto distributions [17]. Additionally, the log-concave CDF class includes subsets of the Weibull, Gamma, Beta, Power Function, Chi-square, and Chi distributions [18]. The log of the CDF from Student's t-distribution is convex, and thus doesn't fall within the log-concave CDF class.

Solving the integrals in [11] or [12] to determine Hanson-Koopmans TI for any percentile/confidence is not a trivial task, and thus a small Python library was created to quickly calculate the TI*. The default solver uses the secant method to determine the TI for any percentile/confidence level. The approximation from Vangel [12] is used as the initial guess for the root-finding algorithm. There is some choice between which two samples to select when using this method. This work uses the first and last ordered samples when presenting Hanson-Koopmans results, because the first and last samples are used for A-basis values [10, 16].

When the distribution is known or can be identified, analytical expressions can be used to determine a TI. One of the most popular statistical distributions is the Normal distribution. The Normal TI can be calculated with tables in [19], or with the analytical equations expressed in [20]. The TI from a Normal distribution occurs at some factor of sample standard deviations away from the sample mean, where the factor varies with the chosen percentile and confidence level.

V. Example with Failure Strength of Composite Material

A histogram of the failure strength on 18 open hole tensile tests is presented in Fig. 2. The tests were conducted by Tomblin and Seneviratne [21], have a width to hole diameter ratio of $\frac{w}{d} = \frac{4}{3}$, and utilized a layup of $[0/90/0/90/45/45/90/0/90/0]_s$. The failure strength does not appear to follow a common distribution, because the histogram depicts a large gap next to the most likely bin. The individual failure strength values were [66.90, 64.04, 62.88, 65.95, 63.25, 59.60, 65.68, 65.71, 64.05, 68.07, 62.77, 64.71, 60.34, 63.76, 58.66, 60.14, 62.68, 58.82] ksi.

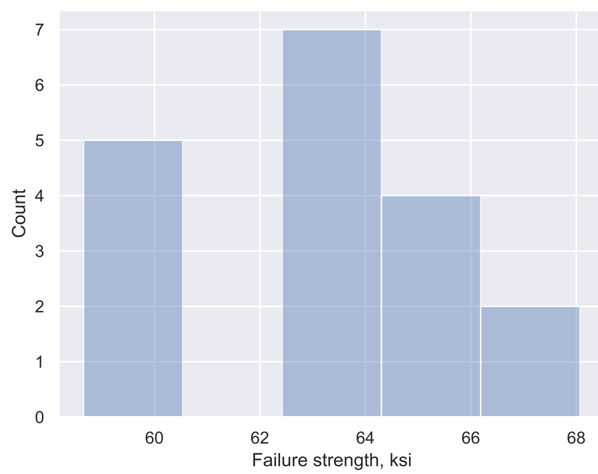
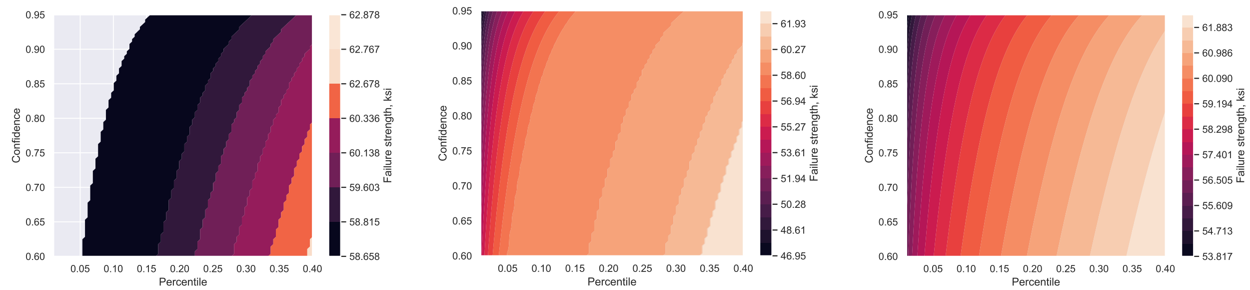


Fig. 2 Histogram of 18 tests of a composite material failure strength.

With the 18 open tensile tests, TIs were calculated using the three different methods. Percentiles were varied from 0.01 to 0.4, and confidence levels were varied from 0.6 to 0.95. The results are shown as contour plots in Fig. 3. The one-order and two-order statistics have discrete steps, while TIs for the Normal distribution are smooth with respect to the percentile/confidence combinations. There is a large region missing for the one-order statistic method, which signifies the percentile/confidence limit with only 18 samples. This region where the one-order statistic exists, the two-order Hanson-Koopmans method produces the exact same result as the one-order statistic. Where the one-order statistic is undefined, the two-order Hanson-Koopmans method extrapolates beyond smallest sample. This phenomenon occurs, because the two-order Hanson-Koopmans method was only intended to be applied in the region where the one-order statistics were undefined. The assumption that the Hanson-Koopmans method applies to the log-concave CDF class is only when extrapolation beyond the smallest sample is needed.

Table 1 shows comparison of the allowable failure stress from the different TI methods for select percentile/confidence levels. Unfortunately, the one-order statistic can not be used to estimate A-basis or B-basis values with only 18 samples. The two-order Hanson-Koopmans TIs presented in the table are more conservative than the TIs from a Normal

*Python code to calculate the Hanson-Koopmans TI is available online at `https://github.com/cjekel/tolerance_interval_py`.



(a) Non-parametric one-order statistic method for a continuous distribution

(b) Hanson-Koopmans two-order statistic method for log-concave CDF class

(c) Normal distribution

Fig. 3 Allowable failure strength from the 18 tests as function of percentile and confidence using (a) one-order non-parametric statistic, (b) two-order Hanson-Koopmans, and (c) Normal distribution. The grid space on (a) denotes the region where the TI does not exist.

Table 1 Comparison of allowables from the different tolerance interval (TI) methods. Note that if the TI is not possible an N/A is used.

P/γ	Any continuous distribution One-order statistic Non-parametric, ksi	Log-concave CDF class Two-order statistic Hanson-Koopmans, ksi	Normal distribution Analytic, ksi
0.01 / 0.95	N/A	47.0	53.8
0.10 / 0.95	N/A	56.6	57.7
0.10 / 0.80	58.7	58.7	58.7
0.20 / 0.85	58.8	58.8	59.9

distribution. The Hanson-Koopmans method can be seen to revert to the more permissive (any continuous distribution) non-parametric method when $P = 0.1, \gamma = 0.8$ and $P = 0.2, \gamma = 0.85$ with 18 samples. This occurs when the bound from the Hanson-Koopmans method occurs within the sample range (as opposed to extrapolating beyond the smallest sample).

VI. Example of Risk Allocation Design Optimization

Consider a single propeller UAV with design specifications listed in Table 2. The UAV follows a design of equal safety margins, where the A-basis allowable (1% with 95% confidence) was used for the skin on the wing and horizontal tail. The A-basis failure strength σ_a from the previous 18 tension tests is 46.95 ksi using the Hanson-Koopmans method. The thickness of the wing and tail skins were varied, such that each component was fully stressed. The probability of failure of the system can then be estimated assuming the failure of each component is independent. Since the Wing and tail are fully stressed, the probability of failure of the wing and tail are each 1%. Then neglecting other components in the design, the system probability of failure P_f is approximately 2%,

$$P_f = 1 - (1 - P_w)(1 - P_t) \tag{7}$$

$$= 1 - (1 - 0.01)(1 - 0.01) \tag{8}$$

$$\approx 0.02 \tag{9}$$

where P_w and P_t are the allowable percentiles from the wing and horizontal tail. It should be stated that it is impossible to know the true probability of failure due to a large number of unknown uncertainties. Despite not knowing the true probability of failure, [1] demonstrated that an aircraft design could be improved with unequal safety factors.

Following similar assumptions used by [1], the weight of the wing and tail become inversely proportional to the

Table 2 Specifications of UAV.

Description	Value
Takeoff weight	15 lb
Wing weight	1.35 lb
Horizontal tail weight	0.3 lb
Wingspan	9 ft

allowable failure stress. This relationship can be expressed as

$$\frac{\sigma_a}{\sigma_n} = \frac{w_n}{w_i} \tag{10}$$

where σ_a is the A-basis failure strength, σ_n is the new allowable failure strength, w_n is the new component weight, and w_i is the initial component weight. This relationship assumes that changes in the weight result from changes in material thickness of the skin on the wing and horizontal tail. The weight of internal components within the skin is assumed to be negligible when compared to the weight of the skin.

An RBDO problem can be setup to improve from the design using equal safety margin. The optimization problem to minimize weight is defined as

$$\min w = w_w(p_w, \gamma_w) + w_t(p_t, \gamma_t) \tag{11}$$

$$\text{such that: } P_f \leq 0.02 \tag{12}$$

$$\gamma_w = \gamma_t = 0.95 \tag{13}$$

where design variables (P_w and P_t) are the percentile of the allowable failure stress for the wing and tail designs. The Hanson-Koopmans method is used to estimate the allowable stress with 95% confidence. The constraint ensures that the new probabilistic design has the same probability of failure as the previous design.

Figure 4 shows a contour plot of the objective from the RBDO with respect to the wing and tail allowable percentiles. These contours show that the overall weight is more sensitive to the wing allowable, which is the heavier of the two components. The optimization problem can be reduced to a one dimensional search along the probability of failure constraint boundary, as illustrated in Fig. 5. The curve represents how changes in the wing allowable have on the total weight, and all points on the curve have the same probability of failure. The wing and tail of the RBDO optimum offers some weight improvement when compared with the previous design using equal safety margin.

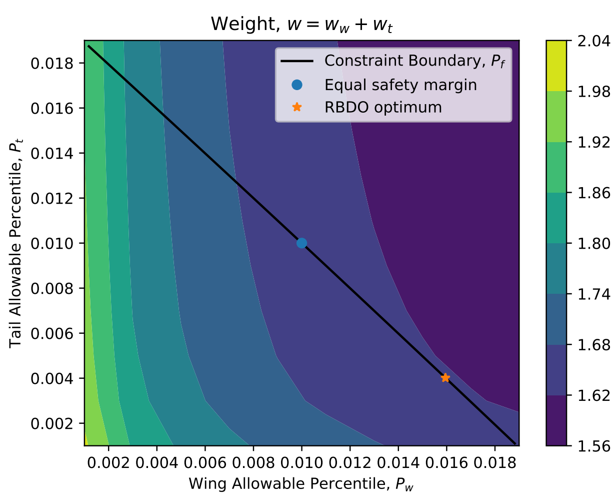


Fig. 4 Contour plot of the weight with respect to wing and horizontal tail allowables from the Hanson-Koopmans method. The constraint boundary represents a 2% probability of failure.

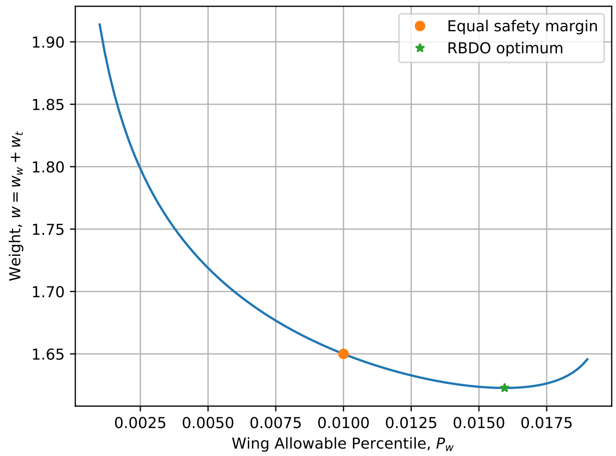


Fig. 5 One dimensional plot along the constraint boundary shows the new RBDO optimum and the location of the previous UAV design, where the wing and tail used the A-basis allowable from the Hanson-Koopmans method.

The resulting two different designs are described in Table 3. One design required the wing and tail to be designed with the same A-basis allowable (and effective safety factor), while the RBDO optimum allocates risk between the wing and tail respectively. The RBDO optimum allocated risk from the lighter component (horizontal tail) to the heavier component (UAV wing), in order to minimize the total weight.

Table 3 Comparison of the UAV specifications from the previous equal safety margin design and the new RBDO optimal design.

TI method	Equal safety margin Hanson-Koopmans	RBDO optimum Hanson-Koopmans	RBDO optimum Normal Distribution
Probability of Failure P_f	0.020	0.020	0.020
Wing allowable percentile P_w	0.010	0.016	0.017
Tail allowable percentile P_t	0.010	0.004	0.003
Wing allowable stress, ksi	46.95	48.77	54.56
Tail allowable stress, ksi	46.95	43.58	52.36
Wing weight w_w, lb	1.35	1.30	1.16
Tail weight w_t, lb	0.30	0.32	0.27
Weight $w = w_w + w_t$, lb	1.65	1.62	1.43

The RBDO optimum reduced the weight by 2% (or 0.03 lb), for the same probability of failure as a design where both components utilized the same A-basis allowable failure strength. This is perhaps an underwhelming result when compared to the 15 lb takeoff weight of the UAV. The more interesting aspect is that the RBDO allocates risk depending upon the component weight. The wing allowable increased from 1% to 1.6%, while the tail allowable reduced from 1% to 0.4%. Additionally, the RBDO was performed assuming that the failure stress belonged to the log-concave CDF class of distributions, rather than assuming the failure stress belonged to a single continuous distribution.

Similar risk allocation results occur if TIs from the Normal distribution are used instead of the Hanson-Koopmans method. The last column in Table 3 shows the results which assume the composite failure strength followed a Normal distribution. The wing allowable was increased from 1% to 1.7%, instead of 1.6% when the Hanson-Koopmans method was used. The tail allowable was decreased from 1% to 0.3%, instead of 0.4% from the Hanson-Koopmans method. This appears to show that the risk allocation was not sensitive to using the Normal distribution or the log-concave CDF class of distributions.

The optimum using the Normal distribution TI was about 12% lighter than using the Hanson-Koopmans method.

Although the lighter design is not as safe, especially if the failure strength does not actually follow a Normal distribution. The wing and tail allowable stresses of 54.56 ksi and 52.36 ksi correspond to the 6.3 and 3.8 percentiles, if the Hanson-Koopmans method was used to 95% confidence. If the failure strength did not follow the Normal distribution, but rather followed the log-concave CDF class, then the small 2% becomes a much larger 10% probability of failure. This may still be acceptable risk for a UAV rather than a manned vehicle.

VII. Conclusion

Typical RBDO problems assume either that the true distribution is known or can be identified, however identifying the distribution can be difficult in practice. The failure strength from 18 replicate tests on the same composite were presented as an example where the true distribution is unknown. Non-parametric tolerance intervals (TI) were disused for cases where the true distribution is not known. The Hanson-Koopmans non-parametric method assumes the true distribution to belong to the log-concave CDF class of distributions, which accounts for the epistemic uncertainty in the unknown distribution. The Hanson-Koopmans method was then used in an RBDO problem to minimize weight of the wing and horizontal tail for a UAV. The RBDO solution allocated risk between the two components of substantially unequally weight. Risk allocation resulted in a lighter design for the same probability of failure as a design with equal safety margins on the wing and horizontal tail. The proportion of risk allocation appeared insensitive to the TI method, as the Hanson-Koopmans RBDO results were similar to assuming the failure strength followed a Normal distribution.

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